

# Nonlinear Fractional Backward Doubly Stochastic Differential Equations with Hurst Parameter in $(1/2, 1)$

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## Abstract

We first state a special type of Itô formula involving stochastic integrals of both standard and fractional Brownian motions. Then we use Doss-Sussman transformation to establish the link between backward doubly stochastic differential equations, driven by both standard and fractional Brownian motions, and backward stochastic differential equations, driven only by standard Brownian motions. Following the same technique, we further study associated nonlinear stochastic partial differential equations driven by fractional Brownian motions and partial differential equations with stochastic coefficients.

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## 1 Introduction

The theory of backward stochastic differential equations (BSDEs) and that of fractional Brownian motion (fBm) had developed simultaneously in their own separated directions for many years until Bender [2] gave an explicit solution to a linear BSDE driven by fBm in 2005. In 2009 Hu and Peng [10] stated a more general theory on fractional BSDEs by using the so-called quasi-expectation, but their case is still limited. The non-semimartingale property of the fBm (except the case of Hurst parameter  $H = 1/2$ , where it becomes a Brownian motion) makes it thorny to handle. Being not a semimartingale means there is no martingale representation theory for the fBm, which is crucial in the general BSDE theory (see the pioneering work of Pardoux and Peng [15]). In Jing and León [11], we tried to combine the fBm and BSDEs in another way: We transformed a semilinear backward doubly stochastic differential equation (BDSDE) driven by both a standard and a fractional Brownian motions with  $H \in (0, 1/2)$  into a BSDE without integral with respect to the fBm, which turns out easier to deal with. The integral w.r.t. the fBm in the BDSDE was interpreted in the sense of the extended divergence operator and the integral was supposed to be linear w.r.t. the solution process. This allowed to apply the efficient tool of nonanticipating Girsanov transformation, developed in Buckdahn [4]. However, this method is restricted to semilinear BDSDEs.

In this paper, we deal with BDSDEs for which the integrand of the integral with respect to the fBm is not necessarily linear with the solution process, and the Hurst coefficient  $H$  is supposed to belong to the

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interval  $(1/2, 1)$ . Unlike the more irregular case  $H < 1/2$ , the stochastic integrals with respect to an fBm with  $H > 1/2$  can be defined in different ways. So they can be defined with the help of the divergence operator in the frame of the Malliavin calculus, see Decreusefont and Üstünel [8] and Alòs *et al.* [1] (Notice that the Wick-Itô integral defined in Duncan *et al.* [9] coincides with the first one). They can also be defined pathwise as generalized Riemann-Stieltjes integral (see Zähle [24] and [25]) or with the help of the rough path theory (see Coutin and Qian [7]). For a complete list of references we refer to the two books by Biagini *et al.* [3] and Mishura [13].

Our approach to BDSDEs with an fBm is inspired by the work of Buckdahn and Ma [5]. In their study of stochastic PDEs driven by a Brownian motion  $B$  the authors of [5] used BDSDEs driven by  $B$  as well as an independent Brownian motion; the integral with respect to  $B$  is interpreted in Stratonovich sense. This allowed the application of the Doss-Sussman transformation in order to transform the BDSDE into a BSDE without integral with respect to  $B$ . On the other hand, the pathwise integral with respect to the fBm plays a role which is comparable with that of the Stratonovich integral in the classical theory. Nualart and Răşcanu [14] used the pathwise integral to solve (forward) stochastic differential equations driven by an fBm. For some technical reasons (such as the lack of Hölder continuity, see Remark 4.12), we shall make use of the Russo-Vallois integral developed by Russo and Vallois in a series of papers ([21], [21], [21], *etc.*). Under standard assumptions which allow to apply the Doss-Sussman transformation, we associate the BDSDE driven by both a standard Brownian motion  $W$  and an fBm  $B$ ,

$$U_s^{t,x} = \Phi(X_0^{t,x}) + \int_0^s f(r, X_r^{t,x}, U_r^{t,x}, V_r^{t,x})dr + \int_0^s g(U_r^{t,x})dB_r - \int_0^s V_r^{t,x} \downarrow dW_r, \quad s \in [0, t],^1 \quad (1.1)$$

with the BSDE driven only by the Brownian motion  $W$ ,

$$Y_s^{t,x} = \Phi(X_0^{t,x}) + \int_0^s \tilde{f}(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x})dr - \int_0^s Z_r^{t,x} \downarrow dW_r, \quad s \in [0, t]. \quad (1.2)$$

Here  $\tilde{f}$  will be specified in Section 4; it is a driver with quadratic growth in  $z$ . We point out that the classical BDSDEs were first studied by Pardoux and Peng [16] and our BSDE (1.2) is a quadratic growth BSDE, which was studied first by Kobylanski [12]. In the works of Pardoux and Peng [16] and Buckdahn and Ma [5], i.e., when the Hurst parameter  $H = 1/2$ , one can solve the BDSDE directly and get the square integrability of the solution process. However, in the fractional case ( $H \neq 1/2$ ), to our best knowledge, there does not exist a direct way to solve the BDSDE (1.1), and as it turns out in Theorem 4.7, we can only get that the conditional expectation of  $\int_0^t |Z_r^{t,x}|^2 dr$  is bounded by an a.s. finite process. This is also the reason that instead of using the space of square integrable processes, we use the space of a.s. conditionally square integrable processes (see the definition of the space  $\mathcal{H}_T^2(\mathbb{R}^d)$  in Section 2).

A celebrated contribution of the BSDE theory consists in giving a form of probabilistic interpretation, nonlinear Feynman-Kac formula, to the solutions of PDEs (see, for instance, Peng [18], Pardoux and Peng [17]). As indicated in Kobylanski [12], the quadratic growth BSDE (1.2) is connected to the semilinear parabolic PDE

$$\begin{cases} \frac{\partial u}{\partial t}(t, x) &= \mathcal{L}u(t, x) - \tilde{f}(t, x, u(t, x), \sigma(x)^T \frac{\partial}{\partial x} u(t, x)), & (t, x) \in (0, T) \times \mathbb{R}^n; \\ u(0, x) &= \Phi(x), & x \in \mathbb{R}^n, \end{cases} \quad (1.3)$$

where  $\mathcal{L}$  is the infinitesimal operator of a Markov process. Hence, it is natural for us to consider the form of equation (1.3) after Doss-Sussman transformation and we prove that it becomes the following semilinear SPDE

$$\begin{cases} du(t, x) &= [\mathcal{L}u(t, x) - f(t, x, u(t, x), \sigma(x)^T \frac{\partial}{\partial x} u(t, x))] dt + g(u(t, x))dB_t, & (t, x) \in (0, T) \times \mathbb{R}^n; \\ u(0, x) &= \Phi(x), & x \in \mathbb{R}^n. \end{cases} \quad (1.4)$$

We emphasize that this paper can not be considered as a generalization of [11]. The reason is that, firstly, the Hurst parameters are distinct; secondly, the stochastic integrals with respect to the fBm are of different types.

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<sup>1</sup>  $\int_0^t \downarrow dW_s$  indicates that the integral is considered as the Itô backward one.

We organize the paper as follows. In section 2, we recall some basic facts about the fractional Brownian motion, we give the general framework of our work, and we recall the definition of the backward Russo-Vallois integral as well as some of its properties. In section 3 we prove a type of Itô formula involving integrals with respect to both standard and fractional Brownian motions, which will play an important role in the following sections. We perform a Doss-Sussman transformation in Section 4 to transform a nonlinear BDSDE (1.1) into a BSDE (1.2) and show the relationship between their solutions. In particular, we show that BSDE (1.2) has a unique solution  $(Y^{t,x}, Z^{t,x})$ , and the couple of processes  $(U^{t,x}, V^{t,x})$  associated with  $(Y^{t,x}, Z^{t,x})$  by the inverse Doss-Sussman transformation is the unique solution of BDSDE (1.1). Finally, the stochastic PDE associated with BDSDE (1.1) is briefly discussed in Section 5.

## 2 Preliminaries

### 2.1 Fractional Brownian Motion and General Setting

In this subsection we recall some basic results on the fBm and the related setting. For a more complete overview of the theory of fBm, we refer the reader to Biagini *et al.* [3] and Mishura [13].

Let  $(\Omega', \mathcal{F}', \mathbb{P}')$  be a classical Wiener space with time horizon  $T > 0$ , i.e.,  $\Omega' = C_0([0, T]; \mathbb{R})$  denotes the set of real-valued continuous functions starting from zero at time zero, endowed with the topology of the uniform convergence,  $\mathcal{B}(\Omega')$  is the Borel  $\sigma$ -algebra on  $\Omega'$  and  $\mathbb{P}'$  is the unique probability measure on  $(\Omega, \mathcal{B}(\Omega'))$  with respect to which the coordinate process  $W_t^0(\omega') = \omega'(t)$ ,  $t \in [0, T]$ ,  $\omega' \in \Omega'$  is a standard Brownian motion. By  $\mathcal{F}'$  we denote the completion of  $\mathcal{B}(\Omega')$  by all  $\mathbb{P}'$ -null sets in  $\Omega'$ . Given  $H \in (1/2, 1)$ , we define

$$B_t = \int_0^t K_H(t, s) dW_s^0, \quad t \in [0, T],$$

where  $K_H$  is the kernel of the fBm with parameter  $H \in (1/2, 1)$ :

$$K_H(t, s) = C_H s^{1/2-H} \int_s^t u^{H-1/2} (u-s)^{H-3/2} du,$$

with  $C_H = \sqrt{\frac{H}{(2H-1)\beta(2-2H, H-1/2)}}$ . It is well known that such defined process  $B$  is a one-dimensional fBm, i.e., it is a Gaussian process with zero mean and covariance function

$$R_H(t, s) := \mathbb{E}[B_t B_s] = \frac{1}{2} (t^{2H} + s^{2H} - |t-s|^{2H}), \quad s, t \in [0, T].$$

We let  $\{W_t : 0 \leq t \leq T\}$  be the coordinate process on the classical Wiener space  $(\Omega'', \mathcal{F}'', \mathbb{P}'')$  with  $\Omega'' = C_0([0, T]; \mathbb{R}^d)$ , which is a  $d$ -dimensional Brownian motion with respect to the Wiener measure  $\mathbb{P}''$ . We put  $(\Omega, \mathcal{F}^0, \mathbb{P}) = (\Omega', \mathcal{F}', \mathbb{P}') \otimes (\Omega'', \mathcal{F}'', \mathbb{P}'')$  and let  $\mathcal{F} = \mathcal{F}^0 \vee \mathcal{N}$ , where  $\mathcal{N}$  is the class of the  $\mathbb{P}$ -null sets. We denote again by  $B$  and  $W$  the canonical extensions of the fBm  $B$  and of the Brownian motion  $W$  from  $\Omega'$  and  $\Omega''$ , respectively, to  $\Omega$ .

We let  $\mathcal{F}_{[t, T]}^W = \sigma\{W_T - W_s, t \leq s \leq T\} \vee \mathcal{N}$ ,  $\mathcal{F}_t^B = \sigma\{B_s, 0 \leq s \leq t\} \vee \mathcal{N}$ , and  $\mathcal{G}_t = \mathcal{F}_{[t, T]}^W \vee \mathcal{F}_t^B$ ,  $t \in [0, T]$ . Let us point out that  $\mathcal{F}_{[t, T]}^W$  is decreasing and  $\mathcal{F}_t^B$  is increasing in  $t$ , but  $\mathcal{G}_t$  is neither decreasing nor increasing. We denote the family of  $\sigma$ -fields  $\{\mathcal{G}_t\}_{0 \leq t \leq T}$  by  $\mathbb{G}$ . Moreover, we also introduce the backward filtrations  $\mathbb{H} = \{\mathcal{H}_t = \mathcal{F}_{[t, T]}^W \vee \mathcal{F}_T^B\}_{t \in [0, T]}$  and  $\mathbb{F}^W = \{\mathcal{F}_{[t, T]}^W\}_{t \in [0, T]}$ .

Finally, we denote by  $C(\mathbb{H}, [0, T]; \mathbb{R}^m)$  the space of the  $\mathbb{R}^m$ -valued continuous processes  $\{\varphi_t, t \in [0, T]\}$  such that  $\varphi_t$  is  $\mathcal{H}_t$ -measurable,  $t \in [0, T]$ , and  $\mathcal{M}^2(\mathbb{F}^W, [0, T]; \mathbb{R}^m)$  the space of the  $\mathbb{R}^m$ -valued square-integrable processes  $\{\psi_t, t \in [0, T]\}$  such that  $\psi_t$  is  $\mathcal{F}_{[t, T]}^W$ -measurable,  $t \in [0, T]$ . Let  $\mathcal{H}_T^\infty(\mathbb{R})$  be the set of  $\mathbb{H}$ -progressively measurable processes which are almost surely bounded by some real-valued  $\mathcal{F}_T^B$ -measurable random variable, and let  $\mathcal{H}_T^2(\mathbb{R}^d)$  denote the set of all  $\mathbb{R}^d$ -valued  $\mathbb{H}$ -progressively measurable processes  $\gamma = \{\gamma_t : t \in [0, T]\}$  such that  $\mathbb{E} \left[ \int_0^T |\gamma_t|^2 dt | \mathcal{F}_T^B \right] < +\infty$ ,  $\mathbb{P}$ -a.s.

## 2.2 Russo-Vallois Integral

In a series of papers ([20], [21], [22], *etc.*), Russo and Vallois defined new types of stochastic integrals, namely forward, backward and symmetric integrals, which are extensions of the classical Riemann-Stieltjes integral, and in fact these three integrals coincide, when the integrator is a fBm with Hurst parameter  $H \in (1/2, 1)$ . Here we will mainly use the backward Russo-Vallois integral in this paper. It turns out to be a convenient definition for stochastic integral with respect to our fBm  $B$ .

Let us recall some results by Russo and Vallois which we will use later. In what follows, we make the convention that all continuous processes  $\{X_t, t \in [0, T]\}$  are extended to the whole line by putting  $X_t = X_0$ , for  $t < 0$ , and  $X_t = X_T$ , for  $t > T$ .

**Definition 2.1.** *Let  $X$  and  $Y$  be two continuous processes. For  $\varepsilon > 0$ , we set*

$$I(\varepsilon, t, X, dY) \triangleq \frac{1}{\varepsilon} \int_0^t X(s)(Y(s) - Y(s - \varepsilon))ds,$$

$$C_\varepsilon(X, Y)(t) \triangleq \frac{1}{\varepsilon} \int_0^t (X(s) - X(s - \varepsilon))(Y(s) - Y(s - \varepsilon))ds, \quad t \in [0, T].$$

*Then the backward Russo-Vallois integral is defined as the uniform limit in probability as  $\varepsilon \rightarrow 0^+$ , if the limit exists. The generalized bracket  $[X, Y]$  is the uniform limit in probability of  $C_\varepsilon(X, Y)$  as  $\varepsilon \rightarrow 0^+$  (of course, again under the condition of existence).*

We recall that (cf. Protter [19]) a sequence of processes  $(H_n; n \geq 0)$  converges to a process  $H$  uniformly in probability if

$$\lim_{n \rightarrow \infty} \mathbb{P} \left( \sup_{t \in [0, T]} |H_n(t) - H(t)| > \alpha \right) = 0 \quad \text{for every } \alpha > 0.$$

In [22] (Theorem 2.1) Russo and Vallois derived the Itô formula for the backward Russo-Vallois integral.

**Theorem 2.2.** *Let  $f \in C^2(\mathbb{R})$  and  $X$  be a continuous process admitting the generalized bracket, i.e.,  $[X, X]$  exists in the sense of Definition 2.1. Then for every  $t \in [0, T]$ , the backward Russo-Vallois integral  $\int_0^t f'(X(s))dX(s)$  exists and*

$$f(X(t)) = f(X(0)) + \int_0^t f'(X(s))dX(s) - \frac{1}{2} \int_0^t f''(X(s))d[X, X](s),$$

*for every  $t \geq 0$ .*

We list some properties of Russo-Vallois integral, which will be used later in this paper.

**Proposition 2.3.** (1). *If  $X$  is a finite quadratic variation process (i.e.,  $[X, X]$  exists and  $[X, X]_T < +\infty$ ,  $\mathbb{P}$ -a.s.) and  $Y$  is a zero quadratic variation process (i.e.,  $[Y, Y]$  exists and equals to zero), then the mutual generalized bracket  $[X, Y]$  exists and vanishes,  $\mathbb{P}$ -a.s.*

(2). *If  $X$  and  $Y$  have  $\mathbb{P}$ -a.s. Hölder continuous paths with order  $\alpha$  and  $\beta$ , respectively, such that  $\alpha > 0$ ,  $\beta > 0$  and  $\alpha + \beta > 1$ , then  $[X, Y] = 0$ .*

(3). *We assume that  $X$  and  $Y$  are continuous and admit a mutual bracket. Then, for every continuous process  $\{H(s) : s \in [0, T]\}$ ,*

$$\int_0^\cdot H(s)dC_\varepsilon(X, Y)(s) \quad \text{converges to} \quad \int_0^\cdot H(s)d[X, Y](s).$$

The following proposition, which can be found in Russo and Vallois [20], states the relationship between the Young integral (see Young [23]) and the backward Russo-Vallois integral.

**Proposition 2.4.** *Let  $X, Y$  be two real processes with paths being  $\mathbb{P}$ -a.s. in  $C^\alpha$  and  $C^\beta$ , respectively, with  $\alpha > 0$ ,  $\beta > 0$  and  $\alpha + \beta > 1$ . Then the backward Russo-Vallois integral  $\int_0^\cdot YdX$  coincides with the Young integral  $\int_0^\cdot Yd^{(y)}X$ .*

### 3 A Generalized Itô Formula

In this section we state a generalized Itô formula involving an Itô backward integral with respect to the Brownian motion  $W$  and the Russo-Vallois integral with respect to the fBm  $B$ . It will play an important role in our paper. It is noteworthy that this Itô formula corresponds to Lemma 1.3 in the paper of Pardoux and Peng [16] for the case of an fBm with Hurst parameter  $H = 1/2$ , i.e., when  $B$  is a Brownian motion.

**Theorem 3.1.** *Let  $\alpha \in C(\mathbb{H}, [0, T]; \mathbb{R})$  be a process of the form*

$$\alpha_t = \alpha_0 + \int_0^t \beta_s ds + \int_0^t \gamma_s \downarrow dW_s, \quad t \in [0, T],$$

where  $\beta$  and  $\gamma$  are  $\mathbb{H}$ -adapted processes and  $\mathbb{P}\{\int_0^T |\beta_s| ds < \infty\} = 1$  and  $\mathbb{P}\{\int_0^T |\gamma_s|^2 ds < +\infty\} = 1$ , respectively. Suppose that  $F \in C^2(\mathbb{R} \times \mathbb{R})$ . Then the Russo-Vallois integral  $\int_0^t \frac{\partial F}{\partial y}(\alpha_s, B_s) dB_s$  (defined as the uniform limit in probability of  $\frac{1}{\varepsilon} \int_0^t (B_s - B_{s-\varepsilon}) \frac{\partial F}{\partial y}(\alpha_s, B_s) ds$ ) exists for  $0 \leq t \leq T$ , and it holds that,  $\mathbb{P}$ -almost surely, for all  $0 \leq t \leq T$ ,

$$\begin{aligned} F(\alpha_t, B_t) = & F(\alpha_0, 0) + \int_0^t \frac{\partial F}{\partial x}(\alpha_s, B_s) \beta_s ds + \int_0^t \frac{\partial F}{\partial x}(\alpha_s, B_s) \gamma_s \downarrow dW_s + \int_0^t \frac{\partial F}{\partial y}(\alpha_s, B_s) dB_s \\ & - \frac{1}{2} \int_0^t \frac{\partial^2 F}{\partial x^2}(\alpha_s, B_s) |\gamma_s|^2 ds. \end{aligned} \quad (3.1)$$

*Proof: Step 1.* First we suppose  $F \in C_b^2(\mathbb{R} \times \mathbb{R})$  (i.e., the function  $F$  is twice continuously differentiable and has bounded derivatives of order less than or equal to two) and there is a positive constant  $C$  such that  $\int_0^T |\beta_s| ds \leq C$  and  $\int_0^T |\gamma_s|^2 ds \leq C$ . It is direct to check that

$$F(\alpha_t, B_t) - F(\alpha_0, 0) = \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \int_0^t (F(\alpha_s, B_s) - F(\alpha_{s-\varepsilon}, B_{s-\varepsilon})) ds.$$

For simplicity we put  $\alpha_{a,\varepsilon,s} \triangleq \alpha_s - a(\alpha_s - \alpha_{s-\varepsilon})$  and  $B_{a,\varepsilon,s} \triangleq B_s - a(B_s - B_{s-\varepsilon})$ , for any  $a \in [0, 1]$ ,  $s \in [0, T]$ ,  $\varepsilon > 0$ . We have

$$\begin{aligned} & F(\alpha_s, B_s) - F(\alpha_{s-\varepsilon}, B_{s-\varepsilon}) \\ = & (\alpha_s - \alpha_{s-\varepsilon}) \frac{\partial F}{\partial x}(\alpha_s, B_s) + (B_s - B_{s-\varepsilon}) \frac{\partial F}{\partial y}(\alpha_s, B_s) - (\alpha_s - \alpha_{s-\varepsilon})^2 \int_0^1 \frac{\partial^2 F}{\partial x^2}(\alpha_{a,\varepsilon,s}, B_{a,\varepsilon,s}) (1-a) da \\ & - 2(\alpha_s - \alpha_{s-\varepsilon})(B_s - B_{s-\varepsilon}) \int_0^1 \frac{\partial^2 F}{\partial x \partial y}(\alpha_{a,\varepsilon,s}, B_{a,\varepsilon,s}) (1-a) da \\ & - (B_s - B_{s-\varepsilon})^2 \int_0^1 \frac{\partial^2 F}{\partial y^2}(\alpha_{a,\varepsilon,s}, B_{a,\varepsilon,s}) (1-a) da. \end{aligned} \quad (3.2)$$

By applying the stochastic Fubini theorem, we get that

$$\begin{aligned} & \frac{1}{\varepsilon} \int_0^t (\alpha_s - \alpha_{s-\varepsilon}) \frac{\partial F}{\partial x}(\alpha_s, B_s) ds \\ = & \frac{1}{\varepsilon} \int_0^t \left( \int_{s-\varepsilon}^s \beta_r dr + \int_{s-\varepsilon}^s \gamma_r \downarrow dW_r \right) \frac{\partial F}{\partial x}(\alpha_s, B_s) ds \\ = & \frac{1}{\varepsilon} \int_0^t \int_r^{(r+\varepsilon) \wedge t} \beta_r \frac{\partial F}{\partial x}(\alpha_s, B_s) ds dr + \frac{1}{\varepsilon} \int_0^t \int_r^{(r+\varepsilon) \wedge t} \gamma_r \frac{\partial F}{\partial x}(\alpha_s, B_s) ds \downarrow dW_r. \end{aligned}$$

Since  $\frac{1}{\varepsilon} \int_r^{(r+\varepsilon) \wedge t} \frac{\partial F}{\partial x}(\alpha_s, B_s) ds$  is  $\mathcal{H}_t$ -measurable and converges to  $\frac{\partial F}{\partial x}(\alpha_r, B_r)$  when  $\varepsilon \rightarrow 0$ , it follows that,

thanks to the continuity of  $\frac{\partial F}{\partial x}(\alpha_s, B_s)$ ,

$$\begin{aligned} & \lim_{\varepsilon \rightarrow 0} \sup_{t \in [0, T]} \left| \int_0^t \beta_r \left( \frac{1}{\varepsilon} \int_r^{(r+\varepsilon) \wedge t} \frac{\partial F}{\partial x}(\alpha_s, B_s) ds - \frac{\partial F}{\partial x}(\alpha_r, B_r) \right) dr \right| \\ & \leq \lim_{\varepsilon \rightarrow 0} \sup_{r \in [0, T]} \left| \frac{1}{\varepsilon} \int_r^{(r+\varepsilon) \wedge t} \frac{\partial F}{\partial x}(\alpha_s, B_s) ds - \frac{\partial F}{\partial x}(\alpha_r, B_r) \right| \int_0^T |\beta_r| dr = 0, \quad \text{in probability.} \end{aligned}$$

Thus, in virtue of the boundedness of  $\frac{\partial F}{\partial x}$ , by the Dominated Convergence Theorem,

$$\begin{aligned} & \lim_{\varepsilon \rightarrow 0} \mathbb{E} \left[ \sup_{t \in [0, T]} \left| \int_0^t \gamma_r \left( \frac{1}{\varepsilon} \int_r^{(r+\varepsilon) \wedge t} \frac{\partial F}{\partial x}(\alpha_s, B_s) ds - \frac{\partial F}{\partial x}(\alpha_r, B_r) \right) \downarrow dW_r \right|^2 \middle| \mathcal{F}_T^B \right] \\ & \leq \lim_{\varepsilon \rightarrow 0} \mathbb{E} \left[ \sup_{r \in [0, T]} \left| \frac{1}{\varepsilon} \int_r^{(r+\varepsilon) \wedge t} \frac{\partial F}{\partial x}(\alpha_s, B_s) ds - \frac{\partial F}{\partial x}(\alpha_r, B_r) \right|^2 \int_0^t |\gamma_r|^2 dr \middle| \mathcal{F}_T^B \right] = 0, \quad \mathbb{P} - \text{a.s.} \end{aligned}$$

(Recall that  $\left( \frac{1}{\varepsilon} \int_r^{(r+\varepsilon) \wedge t} \frac{\partial F}{\partial x}(\alpha_s, B_s) ds - \frac{\partial F}{\partial x}(\alpha_r, B_r) \right)_{r \in [0, T]}$  is  $\mathbb{H}$ -adapted and  $W_\cdot - W_T$  is an  $\mathbb{H}$ -(backward) Brownian motion.) Thus, we get

$$\lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \int_0^t (\alpha_s - \alpha_{s-\varepsilon}) \frac{\partial F}{\partial x}(\alpha_s, B_s) ds = \int_0^t \beta_r \frac{\partial F}{\partial x}(\alpha_r, B_r) dr + \int_0^t \gamma_r \frac{\partial F}{\partial x}(\alpha_r, B_r) \downarrow dW_r, \quad (3.3)$$

uniformly in probability. We notice that the generalized bracket of  $\alpha$  is the same as the classical one, i.e.,  $[\alpha, \alpha]_s = \int_0^s |\gamma_r|^2 dr$ ,  $s \in [0, T]$ . We also have

$$\begin{aligned} & \frac{1}{\varepsilon} \int_0^t (\alpha_s - \alpha_{s-\varepsilon})^2 \int_0^1 \frac{\partial^2 F}{\partial x^2}(\alpha_{a,\varepsilon,s}, B_{a,\varepsilon,s})(1-a) da ds \\ & = \frac{1}{2\varepsilon} \int_0^t (\alpha_s - \alpha_{s-\varepsilon})^2 \frac{\partial^2 F}{\partial x^2}(\alpha_s, B_s) ds + A_{\varepsilon,t}, \quad t \in [0, T], \end{aligned} \quad (3.4)$$

where

$$A_{\varepsilon,t} = \frac{1}{\varepsilon} \int_0^t (\alpha_s - \alpha_{s-\varepsilon})^2 \int_0^1 \left( \frac{\partial^2 F}{\partial x^2}(\alpha_{a,\varepsilon,s}, B_{a,\varepsilon,s}) - \frac{\partial^2 F}{\partial x^2}(\alpha_s, B_s) \right) (1-a) da ds.$$

Proposition 2.3 yields that  $\frac{1}{2\varepsilon} \int_0^t (\alpha_s - \alpha_{s-\varepsilon})^2 \frac{\partial^2 F}{\partial x^2}(\alpha_s, B_s) ds$  converges to  $\int_0^t \frac{\partial^2 F}{\partial x^2}(\alpha_s, B_s) d[\alpha, \alpha]_s$  and the continuity of  $\frac{\partial^2 F}{\partial x^2}$ ,  $\alpha$  and  $B$  implies that  $A_{\varepsilon,t}$  converges to zero. A similar argument shows that

$$\frac{1}{\varepsilon} \int_0^t 2(\alpha_s - \alpha_{s-\varepsilon})(B_s - B_{s-\varepsilon}) \int_0^1 \frac{\partial^2 F}{\partial x \partial y}(\alpha_{a,\varepsilon,s}, B_{a,\varepsilon,s})(1-a) da ds$$

and the term

$$\frac{1}{\varepsilon} \int_0^t (B_s - B_{s-\varepsilon})^2 \int_0^1 \frac{\partial^2 F}{\partial y^2}(\alpha_{a,\varepsilon,s}, B_{a,\varepsilon,s})(1-a) da ds$$

converge in probability, respectively, to

$$2 \int_0^t \frac{\partial^2 F}{\partial x \partial y}(\alpha_s, B_s) d[\alpha, B]_s \quad \text{and} \quad \int_0^t \frac{\partial^2 F}{\partial x \partial y}(\alpha_s, B_s) d[B, B]_s.$$

However, these both latter expressions are zero due to the fact that  $H \in (1/2, 1)$  and Proposition 2.3 (Observe that the fBm has Hölder continuous paths of any positive order less than  $H$  almost surely).

Combining the above results with (3.2), (3.3) and (3.4), we see that

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \int_0^t (B_s - B_{s-\varepsilon}) \frac{\partial F}{\partial y}(\alpha_s, B_s) ds &= F(\alpha_t, B_t) - F(\alpha_0, 0) - \int_0^t \frac{\partial F}{\partial x}(\alpha_s, B_s) \beta_s ds \\ &\quad - \int_0^t \frac{\partial F}{\partial x}(\alpha_s, B_s) \gamma_s \downarrow dW_s + \frac{1}{2} \int_0^t \frac{\partial^2 F}{\partial x^2}(\alpha_s, B_s) |\gamma_s|^2 ds, \end{aligned}$$

uniformly in  $t \in [0, T]$ , in probability. Consequently, the integral  $\int_0^t \frac{\partial F}{\partial y}(\alpha_s, B_s) dB_s$  exists in Russo-Vallois' sense (Recall Definition 2.1) and we get the Itô formula (3.1) for  $F \in C_b^2(\mathbb{R} \times \mathbb{R})$ .

**Step 2.** Now we deal with the general case that  $\mathbb{P}\{\int_0^T |\beta_s| ds < \infty\} = \mathbb{P}\{\int_0^T |\gamma_s|^2 ds < \infty\} = 1$ . For each  $n \in \mathbb{N}$ , we define a sequence of  $\mathbb{H}$ -stopping times by  $\tau_n = \sup\{t \leq T : \int_t^T |\beta_s| ds + \int_t^T |\gamma_s|^2 ds > n\} \vee 0$ , so we know that the processes  $\{\beta_t^n := \beta_t \mathbf{1}_{[\tau_n, T]}(t), t \in [0, T]\}$  and  $\{\gamma_t^n := \gamma_t \mathbf{1}_{[\tau_n, T]}(t), t \in [0, T]\}$  satisfy  $\int_0^T |\beta_s^n| ds \leq n$  and  $\int_0^T |\gamma_s^n|^2 ds \leq n$ ,  $\mathbb{P}$ -a.s. Furthermore, as  $n \rightarrow \infty$ ,  $\tau_n \rightarrow 0$ ,  $\mathbb{P}$ -a.s. We consider the Itô formula for the process  $\alpha_t^n = \alpha_0 + \int_0^t \beta_s^n ds + \int_0^t \gamma_s^n \downarrow dW_s, t \in [0, T]$ . Thanks to the result of **Step 1**, we have that, for every  $n \in \mathbb{N}$ ,

$$\begin{aligned} F(\alpha_t^n, B_t) &= F(\alpha_0, 0) + \int_0^t \frac{\partial F}{\partial x}(\alpha_s^n, B_s) \beta_s^n ds + \int_0^t \frac{\partial F}{\partial y}(\alpha_s^n, B_s) dB_s + \int_0^t \frac{\partial F}{\partial x}(\alpha_s^n, B_s) \gamma_s^n \downarrow dW_s \\ &\quad - \frac{1}{2} \int_0^t \frac{\partial^2 F}{\partial x^2}(\alpha_s^n, B_s) |\gamma_s^n|^2 ds. \end{aligned}$$

Since  $\alpha^n$  converges to  $\alpha$  uniformly in probability on  $[0, T]$ , by letting  $n \rightarrow \infty$  in the above equation, we deduce that

$$\lim_{n \rightarrow \infty} \int_0^t \frac{\partial F}{\partial y}(\alpha_s^n, B_s) dB_s = \lim_{n \rightarrow \infty} \lim_{\varepsilon \rightarrow 0} \int_0^t \frac{1}{\varepsilon} \frac{\partial F}{\partial y}(\alpha_s^n, B_s) (B_s - B_{s-\varepsilon}) ds = \lim_{\varepsilon \rightarrow 0} \int_0^t \frac{1}{\varepsilon} \frac{\partial F}{\partial y}(\alpha_s, B_s) (B_s - B_{s-\varepsilon}) ds$$

exists, it is the Russo-Vallois integral  $\int_0^t \frac{\partial F}{\partial y}(\alpha_s, B_s) dB_s$  and it equals to

$$F(\alpha_t, B_t) - F(\alpha_0, 0) - \int_0^t \frac{\partial F}{\partial x}(\alpha_s, B_s) \beta_s ds - \int_0^t \frac{\partial F}{\partial x}(\alpha_s, B_s) \gamma_s \downarrow dW_s + \frac{1}{2} \int_0^t \frac{\partial^2 F}{\partial x^2}(\alpha_s, B_s) |\gamma_s|^2 ds.$$

**Step 3.** Finally we consider the case  $F \in C^2(\mathbb{R} \times \mathbb{R})$ . We let  $\{\varphi_N\}_{N \in \mathbb{N}}$  be a sequence of infinitely differentiable functions with compact support such that  $\varphi_N(x) = x$  for  $\{(x_1, x_2) : \max(|x_1|, |x_2|) \leq N\}$ ,  $N \in \mathbb{N}$ . We set  $F_N(x) = F(\varphi_N(x))$ , so that  $F_N(x) \in C_b^2(\mathbb{R} \times \mathbb{R})$  for every  $N > 0$ . We notice that  $F_N(\alpha, B)$  and  $F(\alpha, B)$  coincide on the set  $\Omega_N = \{\omega \in \Omega : \sup_{0 \leq s \leq t} |\alpha_s| \leq N, \sup_{0 \leq s \leq t} |B_s| \leq N\}$  and that  $\Omega = \bigcup_{N \geq 1} \Omega_N$ . Due to **Step 1** and **Step 2**, for every  $N$ , we have

$$\begin{aligned} F_N(\alpha_t, B_t) &= F_N(\alpha_0, 0) + \int_0^t \frac{\partial F_N}{\partial x}(\alpha_s, B_s) \beta_s ds + \int_0^t \frac{\partial F_N}{\partial y}(\alpha_s, B_s) dB_s + \int_0^t \frac{\partial F_N}{\partial x}(\alpha_s, B_s) \gamma_s \downarrow dW_s \\ &\quad - \frac{1}{2} \int_0^t \frac{\partial^2 F_N}{\partial x^2}(\alpha_s, B_s) |\gamma_s|^2 ds, \quad t \in [0, T]. \end{aligned}$$

Therefore, for every  $N$ , it holds on  $\Omega_N$  that

$$\begin{aligned} F(\alpha_t, B_t) &= F(\alpha_0, 0) + \int_0^t \frac{\partial F}{\partial x}(\alpha_s, B_s) \beta_s ds + \int_0^t \frac{\partial F}{\partial y}(\alpha_s, B_s) dB_s + \int_0^t \frac{\partial F}{\partial x}(\alpha_s, B_s) \gamma_s \downarrow dW_s \\ &\quad - \frac{1}{2} \int_0^t \frac{\partial^2 F}{\partial x^2}(\alpha_s, B_s) |\gamma_s|^2 ds, \quad t \in [0, T]. \end{aligned}$$

Finally, by letting  $N$  tend to  $+\infty$ , we get the wished result. The proof is complete.  $\blacksquare$

## 4 Doss-Sussman Transformation of Fractional Backward Doubly Stochastic Differential Equations

In what follows, we use the following hypotheses:

(H1) The functions  $\sigma : \mathbb{R}^n \rightarrow \mathbb{R}^{n \times d}$  and  $b : \mathbb{R}^n \rightarrow \mathbb{R}^n$  are Lipschitz continuous.

(H2) The function  $f : [0, T] \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R}$  is Lipschitz in  $(x, y, z) \in \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R}$  with  $|f(t, 0, 0, 0)| \leq C$  uniformly in  $t \in [0, T]$ , the function  $g : \mathbb{R}^n \rightarrow \mathbb{R}$  belongs to  $C_b^3(\mathbb{R})$  and the function  $\Phi$  is bounded.

We fix an arbitrary  $t \in [0, T] \subset \mathbb{R}^+$ . Let  $(X_s^{t,x})_{0 \leq s \leq t}$  be the unique solution of the following stochastic differential equation:

$$\begin{cases} dX_s^{t,x} = -b(X_s^{t,x})ds - \sigma(X_s^{t,x}) \downarrow dW_s, & s \in [0, t], \\ X_t^{t,x} = x. \end{cases} \quad (4.1)$$

Here the stochastic integral  $\int_0^t \cdot \downarrow dW_s$  is again understood as the backward Itô one. The condition (H1) guarantees the existence and uniqueness of the solution  $(X_s^{t,x})_{0 \leq s \leq t}$  in  $\mathcal{M}^2(\mathbb{F}^W, [0, T]; \mathbb{R}^n)$ . Our aim is to study the following backward doubly stochastic differential equation:

$$U_s^{t,x} = \Phi(X_0^{t,x}) + \int_0^s f(r, X_r^{t,x}, U_r^{t,x}, V_r^{t,x})dr + \int_0^s g(U_r^{t,x})dB_r - \int_0^s V_r^{t,x} \downarrow dW_r, \quad s \in [0, t]. \quad (4.2)$$

We emphasize that the integral with respect to the fBm  $B$  is interpreted in the Russo-Vallois sense, while the integral with respect to the Brownian motion  $W$  is the Itô backward one. If  $B$  is a standard Brownian motion, equation (4.2) coincides with the BDSDE which was first studied by Pardoux and Peng [16] in 1994 (apart of a time inversion).

Before we investigate the BDSDE (4.2), we first give the definition of its solution.

**Definition 4.1.** A solution of equation (4.2) is a couple of processes  $(U_s^{t,x}, V_s^{t,x})_{s \in [0, t]}$  such that:

- 1).  $(U_s^{t,x}, V_s^{t,x})_{s \in [0, t]} \in \mathcal{H}_t^\infty(\mathbb{R}) \times \mathcal{H}_t^2(\mathbb{R}^d)$ ;
- 2). The Russo-Vallois integral  $\int_0^s g(U_r^{t,x})dB_r$  is well defined on  $[0, t]$ ;
- 3). Equation (4.2) holds  $\mathbb{P}$ -a.s.

Unlike the classical case, the lack of the semimartingale property of the fBm  $B$  gives an extra difficulty in solving BDSDE (4.2) directly. However, the work of Buckdahn and Ma [5] indicates another possibility to investigate this equation: by using the Doss-Sussman transformation. Let us develop the idea. We denote by  $\eta$  the stochastic flow which is the unique solution of the following stochastic differential equation:

$$\eta(t, y) = y + \int_0^t g(\eta(s, y))dB_s, \quad t \in [0, T], \quad (4.3)$$

where the integral is interpreted in the sense of Russo-Vallois. The solution of such a stochastic differential equation can be written as  $\eta(t, y) = \alpha(y, B_t)$  via Doss transformation (see, for example, Zähle [25]), where  $\alpha(y, z)$  is the solution of the ordinary differential equation

$$\begin{cases} \frac{\partial \alpha}{\partial z}(y, z) = g(\alpha(y, z)), & z \in \mathbb{R}, \\ \alpha(y, 0) = y. \end{cases} \quad (4.4)$$

By the classical PDE theory we know that, for every  $z \in \mathbb{R}$ , the mapping  $y \mapsto \alpha(y, z)$  is a diffeomorphism over  $\mathbb{R}$  and  $(y, z) \mapsto \alpha(y, z)$  is  $C^2$ . In particular, we can define the  $y$ -inverse of  $\alpha(y, z)$  and we denote it by  $h(y, z)$ , such that we have  $\alpha(h(y, z), z) = y$ ,  $(y, z) \in \mathbb{R} \times \mathbb{R}$ . Hence, it follows that

$$\frac{\partial \alpha}{\partial y}(h(y, z), z) \frac{\partial h}{\partial y}(y, z) = 1 \quad \text{and} \quad \frac{\partial \alpha}{\partial z}(h(y, z), z) + \frac{\partial \alpha}{\partial y}(h(y, z), z) \frac{\partial h}{\partial z}(y, z) = 0.$$



Therefore,

$$\frac{\partial h}{\partial z}(y, z) = - \left( \frac{\partial \alpha}{\partial y}(h(y, z), z) \right)^{-1} \frac{\partial \alpha}{\partial z}(h(y, z), z) = - \frac{\partial h}{\partial y}(y, z) \frac{\partial \alpha}{\partial z}(h(y, z), z) = - \frac{\partial h}{\partial y}(y, z) g(y).$$

As a direct consequence we have that also  $\eta(t, \cdot) = \alpha(\cdot, B_t) : \mathbb{R} \rightarrow \mathbb{R}$  is a diffeomorphism and, thus, we can define  $\mathcal{E}(t, y) := \eta(t, \cdot)^{-1}(y) = h(y, B_t)$ ,  $(t, y) \in [0, T] \times \mathbb{R}$ . Moreover, by the Itô formula (Theorem 3.1), we have

$$d\mathcal{E}(t, y) = dh(y, B_t) = \frac{\partial}{\partial z} h(y, B_t) dB_t = - \frac{\partial}{\partial y} \mathcal{E}(t, y) g(y) dB_t, \quad t \in [0, T],$$

i.e., the process  $\mathcal{E}$  satisfies the following equation:

$$\mathcal{E}(t, y) = y - \int_0^t \frac{\partial}{\partial y} \mathcal{E}(s, y) g(y) dB_s, \quad t \in [0, T]. \quad (4.5)$$

Furthermore, we have the following estimates for  $\eta$  and  $\mathcal{E}$ .

**Lemma 4.2.** *There exists a constant  $C > 0$  depending only on the bound of  $g$  and its partial derivatives such that for  $\xi = \eta, \mathcal{E}$ , it holds that,  $P$ -a.s., for all  $(t, y)$ ,*

$$\begin{aligned} |\xi(t, y)| &\leq |y| + C|B_t|, \quad \exp\{-C|B_t|\} \leq \left| \frac{\partial}{\partial y} \xi \right| \leq \exp\{C|B_t|\}, \\ \left| \frac{\partial^2}{\partial y^2} \xi \right| &\leq \exp\{C|B_t|\}, \quad \left| \frac{\partial^3}{\partial y^3} \xi \right| \leq \exp\{C|B_t|\}. \end{aligned}$$

*Proof:* The first three estimates are similar to those in Buckdahn and Ma [5]. So we only prove the last one. For this end we define  $\gamma(\theta, y, z) = \alpha(y, \theta z)$ , for  $(\theta, y, z) \in [0, 1] \times \mathbb{R} \times \mathbb{R}^n$ . It follows from (4.4) that

$$\gamma(\theta, y, z) = y + \int_0^\theta \langle g(\gamma(r, y, z)), z \rangle dr.$$

By differentiating this latter equation, we obtain,

$$\begin{cases} \frac{\partial^4 \gamma}{\partial \theta \partial y^3}(\theta, y, z) = \frac{\partial^3 g}{\partial y^3}(\gamma(\theta, y, z)) z \left( \frac{\partial \gamma}{\partial y}(\theta, y, z) \right)^3 + 3 \frac{\partial^2 g}{\partial y^2}(\gamma(\theta, y, z)) z \frac{\partial \gamma}{\partial y}(\theta, y, z) \frac{\partial^2 \gamma}{\partial y^2}(\theta, y, z) \\ \quad + \frac{\partial g}{\partial y}(\gamma(\theta, y, z)) z \frac{\partial^3 \gamma}{\partial y^3}(\theta, y, z); \\ \frac{\partial^3 \gamma}{\partial y^3}(0, y, z) = 0, \end{cases} \quad (4.6)$$

and from the variation of parameter formula it follows that

$$\begin{aligned} \frac{\partial^3}{\partial y^3} \gamma(1, y, z) &= \int_0^1 \exp \left\{ \int_u^1 \frac{\partial g}{\partial y}(\gamma(v, y, z)) z dv \right\} \left( \frac{\partial^3 g}{\partial y^3}(\gamma(u, y, z)) z \left( \frac{\partial \gamma}{\partial y}(u, y, z) \right)^3 \right. \\ &\quad \left. + 3 \frac{\partial^2 g}{\partial y^2}(\gamma(u, y, z)) z \frac{\partial \gamma}{\partial y}(u, y, z) \frac{\partial^2 \gamma}{\partial y^2}(u, y, z) \right) du. \end{aligned}$$

Thus, by using the first three estimates of this Lemma, we get

$$\left| \frac{\partial^3}{\partial y^3} \eta(t, y) \right| = \left| \frac{\partial^3}{\partial y^3} \alpha(y, B_t) \right| \leq \exp\{C|B_t|\}.$$

Hence we have completed the proof. ■

Lemma 4.2 plays an important role in the rest of the paper thanks to the following lemma.

**Lemma 4.3.** *For any  $C \in \mathbb{R}$ , we have*

$$\mathbb{E} \left[ \exp \left\{ C \sup_{s \in [0, T]} |B_s| \right\} \right] < \infty.$$

*Proof:* The proof is similar as Lemma 2.4 in [11] (and even easier), so we omit it.  $\blacksquare$

We denote by  $\tilde{\Omega}'$  the subspace of  $\Omega'$  such that  $\tilde{\Omega}' = \left\{ \omega' \in \Omega' : \sup_{s \in [0, T]} |B_s| < \infty \right\}$ . It is clear that  $\mathbb{P}'(\tilde{\Omega}') = 1$ .

We let  $(Y^{t,x}, Z^{t,x})$  be the unique solution of the following BSDE:

$$Y_s^{t,x} = \Phi(X_0^{t,x}) + \int_0^s \tilde{f}(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x}) dr - \int_0^s Z_r^{t,x} \downarrow dW_r, \quad (4.7)$$

where

$$\tilde{f}(t, x, y, z) = \frac{1}{\frac{\partial}{\partial y} \eta(t, y)} \left\{ f \left( t, x, \eta(t, y), \frac{\partial}{\partial y} \eta(t, y) z \right) + \frac{1}{2} \text{tr} \left[ z^T \frac{\partial^2}{\partial y^2} \eta(t, y) z \right] \right\}.$$

This BSDE, studied over  $\Omega = \Omega' \times \Omega''$  and driven by the Brownian motion  $W_r(\omega) = W_r(\omega'') = \omega''(r)$ ,  $r \in [0, T]$ , can be interpreted as an  $\omega'$ -pathwise BSDE, i.e., as a BSDE over  $\Omega''$ , considered for every fixed  $\omega' \in \Omega'$ . However, subtleties of measurability make us preferring to consider the BSDE over  $\Omega$ , with respect to the filtration  $\mathbb{H}$ . We point out that the coefficient  $\tilde{f}$  has a quadratic growth in  $z$ , while the terminal value is bounded. BSDEs of this type have been studied by Kobylanski [12]. We state an existence and uniqueness result for this kind of BSDE, but with a slight adaptation to our framework. For this we consider a driving coefficient  $G$  satisfying the following assumptions:

(H3) The coefficient  $G : \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R}^d$  is measurable, for every fixed  $(y, z)$ , progressively measurable, with respect to the backward filtration  $\mathbb{H}$  and  $G$  is continuous in  $(t, y, z)$ ;

(H4) There exists some real-valued  $\mathcal{F}_T^B$ -measurable random variable  $K : \Omega' \rightarrow \mathbb{R}$  such that  $|G(t, y, z)| \leq K(1 + |z|^2)$ .

(H5) There exist real-valued  $\mathcal{F}_T^B$ -measurable random variables  $C > 0$ ,  $\varepsilon > 0$ , and  $\mathcal{F}_T^B \otimes \mathcal{B}([0, T])$ -measurable functions  $k, l_\varepsilon : \Omega' \times [0, T] \rightarrow \mathbb{R}$  such that

$$\left| \frac{\partial G}{\partial z}(t, y, z) \right| \leq k(t) + C|z|, \text{ for all } (t, y, z), \mathbb{P} - \text{a.s.},$$

$$\frac{\partial G}{\partial y}(t, y, z) \leq l_\varepsilon(t) + \varepsilon|z|^2, \text{ for all } (t, y, z), \mathbb{P} - \text{a.s.}$$

**Remark 4.4.** Due to the Lemmata 4.2 and 4.3, the function  $\tilde{f}$  in the equation (4.7) satisfies (H3) – (H5). In particular,  $K = \exp\{C \sup_{t \in [0, T]} |B_t|\}$  for  $C \in \mathbb{R}^+$  appropriately chosen.

Adapting the results by Kobylanski [12] (Theorem 2.3 and Theorem 2.6), we can state the following:

**Theorem 4.5.** Let  $G$  be a driver such that hypotheses (H3)-(H5) hold and let  $\xi$  be a real-valued  $\mathcal{H}_0$ -measurable random variable, which is bounded by a real-valued  $\mathcal{F}_T^B$ -measurable random variable. Then there exists a unique solution  $(Y, Z) \in \mathcal{H}_T^\infty(\mathbb{R}) \times \mathcal{H}_T^2(\mathbb{R}^d)$  of BSDE

$$Y_t = \xi + \int_0^t G(s, Y_s, Z_s) ds - \int_0^t Z_s \downarrow dW_s, \quad t \in [0, T]. \quad (4.8)$$

Moreover, there exists a real-valued  $\mathcal{F}_T^B$ -measurable random variable  $C$  depending only on  $\text{esssup}_{[0, T] \times \Omega''} |Y_t(\omega', \omega'')|$  and  $K$ , such that

$$\mathbb{E} \left[ \int_0^T |Z_s|^2 ds \middle| \mathcal{F}_T^B \right] \leq C, \quad \mathbb{P} - \text{a.s.}$$

**Remark 4.6.** The conditional expectation  $\mathbb{E}[\cdot|\mathcal{F}_T^B]$  is here understood in the generalized sense: if  $\xi$  is a nonnegative  $\mathcal{H}_0$ -measurable random variable,

$$\mathbb{E}[\xi|\mathcal{F}_T^B] := \lim_{n \rightarrow \infty} \uparrow \mathbb{E}[\xi \wedge n|\mathcal{F}_T^B](\leq \infty)$$

is a well defined  $\mathcal{F}_T^B$ -measurable random variable. If  $\xi$  is not nonnegative we decompose  $\xi = \xi^+ - \xi^-$ ,  $\xi^+ = \max\{\xi, 0\}$ ,  $\xi^- = -\min\{\xi, 0\}$  and we put  $\mathbb{E}[\xi|\mathcal{F}_T^B] = \mathbb{E}[\xi^+|\mathcal{F}_T^B] - \mathbb{E}[\xi^-|\mathcal{F}_T^B]$  on  $\{\min\{\mathbb{E}[\xi^+|\mathcal{F}_T^B], \mathbb{E}[\xi^-|\mathcal{F}_T^B]\} < \infty\}$ .

*Proof of Theorem 4.5:* We observe that the Brownian motion  $W$  possesses the (backward) martingale representation with respect to the backward filtration  $\mathbb{H}$ , i.e., given an  $\mathcal{H}_0$ -measurable random variable  $\xi$  such that  $\mathbb{E}[\xi^2|\mathcal{F}_T^B] < \infty$ ,  $\mathbb{P}$ -a.s., there exists a unique process  $\gamma \in \mathcal{H}_T^2(\mathbb{R}^d)$  such that

$$\xi = \mathbb{E}[\xi|\mathcal{F}_T^B] - \int_0^T \gamma_r \downarrow dW_r, \quad \mathbb{P} - \text{a.s.}$$

This martingale representation property allows to show the existence and uniqueness of a solution of (4.8) when  $G$  is of linear growth and Lipschitz in  $(y, z)$ . Combining this with the approach by Kobylanski [12] allows to obtain the result stated in Theorem 4.5.  $\blacksquare$

By using Theorem 4.5, we are now able to characterize more precisely the solution of BSDE (4.7).

**Theorem 4.7.** Under our standard assumptions on the coefficients  $\sigma, b, f$  and  $\Phi$ , BSDE (4.7) admits a unique solution  $(Y^{t,x}, Z^{t,x})$  in  $\mathcal{H}_t^\infty(\mathbb{R}) \times \mathcal{H}_t^2(\mathbb{R}^d)$ . Moreover, there exists a positive increasing process  $\theta \in L^0(\mathbb{H}, \mathbb{R})$  such that  $|Y_s^{t,x}| \leq \theta_s$ ,  $\mathbb{E}[\int_0^\tau |Z_s^{t,x}|^2 ds | \mathcal{H}_\tau] \leq \exp\{\exp\{C \sup_{s \in [0, t]} |B_s|\}\}$ ,  $\mathbb{P}$ -a.s., for all  $\mathbb{H}$ -stopping times  $\tau$  ( $0 \leq \tau \leq t$ ), where  $C$  is a constant chosen in an adequate way. Furthermore, the process  $(Y^{t,x}, Z^{t,x})$  is  $\mathbb{G}$ -adapted.

*Proof:* Due to Theorem 4.5, equation (4.7) has a unique solution  $(Y^{t,x}, Z^{t,x})$  in  $\mathcal{H}_t^\infty(\mathbb{R}) \times \mathcal{H}_t^2(\mathbb{R}^d)$ .

**Step 1.** In order to give the estimates of  $(Y^{t,x}, Z^{t,x})$ , we proceed as in Lemma 5.3 in [5]. In particular, we can show that there exists an increasing positive process  $\theta \in L^0(\mathbb{F}^B, [0, T])$  such that  $\mathbb{P}$ -a.s.,  $|Y_s^{t,x}| \leq \theta_s$ ,  $0 \leq s \leq t \leq T$ . This process  $(\theta_s)$  can be chosen as the solution of the following ordinary differential equation

$$\frac{d\theta_s}{ds} = \exp\{C \sup_{0 \leq r \leq t} |B_r|\}(1 + \theta_s); \quad \theta(0) = |\Phi(X_0^{t,x})|,$$

for some suitably chosen real constant  $C$ , i.e.,

$$\theta_s = (|\Phi(X_0^{t,x})| + 1) \exp\left\{\exp\left\{C \sup_{0 \leq r \leq t} |B_r|\right\} s\right\} - 1.$$

Indeed, for  $M > 0$ , let  $\varphi_M(y)$  be a  $C^\infty$  function such that  $0 \leq \varphi_M \leq 1$ ,  $\varphi_M(y) = 1$  for  $|y| \leq M$  and  $\varphi_M(y) = 0$  for  $|y| \geq M+1$ . Defining a new function  $\tilde{f}^M$  by  $\tilde{f}^M(t, x, y, z) \triangleq \tilde{f}(t, x, y, z)\varphi_M(y)$  we see that the function  $\tilde{f}^M$  also satisfies conditions (H3)-(H5). According to Theorem 4.5, there exists a unique solution  $(Y^{M,t,x}, Z^{M,t,x})$  of equation (4.7) with  $\tilde{f}$  being replaced with  $\tilde{f}^M$ . The stability result in Kobylanski [12] shows that, when  $M \rightarrow +\infty$ , there exists a subsequence of  $Y^{M,t,x}$  converging to  $Y^{t,x}$  uniformly in probability. Therefore, following Buckdahn and Ma's approach and slightly adapted, we only need to prove that  $Y_s^{M,t,x}$  is uniformly bounded by  $\theta_s$ . We apply the Tanaka formula to  $|Y^{M,t,x}|$  to get that

$$|Y_s^{M,t,x}| = |\Phi(X_0^{t,x})| + \int_0^s \text{sgn}(Y_r^{M,t,x}) \tilde{f}^M(r, X_r^{t,x}, Y_r^{M,t,x}, Z_r^{M,t,x}) dr - \int_0^s \text{sgn}(Y_r^{M,t,x}) Z_r^{M,t,x} \downarrow dW_r + L_s - L_0,$$

for a local-time-like process  $L$  such that  $L_t = 0$  and  $L_s = \int_s^t 1_{\{Y_r^{M,t,x}=0\}} dL_r$ . Now we define a new function  $\psi(y)$  by letting  $\psi(y) = e^{2Ky} - 1 - 2Ky - 2K^2y^2$ , for  $y > 0$ , and  $\psi(y) = 0$  for  $y \leq 0$ , where  $K$  is the bound

in Remark 4.4. Then we apply the Itô formula to  $\psi(|Y_s^{M,t,x}| - \theta_s)$  and get

$$\begin{aligned} & \psi(|Y_s^{M,t,x}| - \theta_s) \\ &= \int_0^s \psi'(|Y_r^{M,t,x}| - \theta_r) \operatorname{sgn}(Y_r^{M,t,x}) \left( \tilde{f}^M(r, X_r^{t,x}, Y_r^{M,t,x}, Z_r^{M,t,x}) - \exp\{C \sup_{0 \leq u \leq t} |B_u|\}(1 + \theta_r) \right) dr \\ & \quad - \int_0^s \psi'(|Y_r^{M,t,x}| - \theta_r) \operatorname{sgn}(Y_r^{M,t,x}) Z_r^{M,t,x} \downarrow dW_r + \int_0^s \psi'(|Y_r^{M,t,x}| - \theta_r) dL_r \\ & \quad - \frac{1}{2} \int_0^s \psi''(|Y_r^{M,t,x}| - \theta_r) |Z_r^{M,t,x}|^2 dr \end{aligned} \quad (4.9)$$

The property of  $\psi$  shows that  $\int_0^s \psi'(|Y_r^{M,t,x}| - \theta_r) dL_r = \int_0^s \psi'(-\theta_r) dL_r = 0$ . We also have

$$\begin{aligned} & \int_0^s \psi'(|Y_r^{M,t,x}| - \theta_r) \operatorname{sgn}(Y_r^{M,t,x}) \left( \tilde{f}^M(r, X_r^{t,x}, Y_r^{M,t,x}, Z_r^{M,t,x}) - \exp\{C \sup_{0 \leq u \leq t} |B_u|\}(1 + \theta_r) \right) dr \\ & \leq \int_0^s \psi'(|Y_r^{M,t,x}| - \theta_r) (K(\varphi(Y_r^{M,t,x})|Y_r^{M,t,x}| - \theta_r) + K|Z_r^{M,t,x}|^2) dr. \end{aligned}$$

Since  $\psi'' - 2K\psi' \geq 0$ , we get from equation (4.9) that

$$\begin{aligned} \psi(|Y_s^{M,t,x}| - \theta_s) & \leq K \int_0^s \psi'(|Y_r^{M,t,x}| - \theta_r) (\varphi(Y_r^{M,t,x})|Y_r^{M,t,x}| - \theta_r) dr \\ & \quad - \int_0^s \psi'(|Y_r^{M,t,x}| - \theta_r) \operatorname{sgn}(Y_r^{M,t,x}) Z_r^{M,t,x} \downarrow dW_r. \end{aligned}$$

Thus, we deduce that

$$\mathbb{E}[\psi(|Y_s^{M,t,x}| - \theta_s) | \mathcal{H}_s] \leq \mathbb{E} \left[ K \int_0^s \psi'(|Y_r^{M,t,x}| - \theta_r) (\varphi(Y_r^{M,t,x})|Y_r^{M,t,x}| - \theta_r) dr \middle| \mathcal{H}_s \right].$$

From the definition of  $\psi$  we get that  $\psi'(y) = 2K(\psi(y) + 2K^2y^2)$ . Hence, we have

$$\begin{aligned} & \mathbb{E}[\psi(|Y_s^{M,t,x}| - \theta_s) | \mathcal{H}_s] \\ & \leq \mathbb{E} \left[ \int_0^s 2K^2\psi(|Y_r^{M,t,x}| - \theta_r) (\varphi(Y_r^{M,t,x})|Y_r^{M,t,x}| - \theta_r) + 4K^4(|Y_r^{M,t,x}| - \theta_r)^3 dr \middle| \mathcal{H}_s \right]. \end{aligned}$$

There also exists a  $\tilde{K}$  such that  $y^3 \leq \tilde{K}\psi(y)$ , for all  $y \in \mathbb{R}$ . Consequently, we get that

$$\mathbb{E}[\psi(|Y_s^{M,t,x}| - \theta_s) | \mathcal{H}_s] \leq 2K^2(M + \|\theta\|_{\infty, [0, t]} + 2K^2\tilde{K}) \int_0^s \mathbb{E}[\psi(|Y_r^{M,t,x}| - \theta_r) | \mathcal{H}_s] dr.$$

Finally, the Gronwall inequality shows that  $\psi(|Y_s^{M,t,x}| - \theta_s) = 0$ , for any  $s \in [0, t]$ ,  $\mathbb{P}$ -a.s. Therefore,  $|Y_s^{M,t,x}| \leq \theta_s$ , for any  $s \in [0, t]$ ,  $\mathbb{P}$ -a.s.

**Step 2.** We apply the Itô formula to  $e^{aY_s^{t,x}}$ , with  $a$  being a real-valued  $\mathcal{F}_T^B$ -measurable random variable to be determined later, and we obtain

$$\begin{aligned} e^{aY_s^{t,x}} &= e^{a\Phi(X_0^{t,x})} + \int_0^s ae^{aY_r^{t,x}} \tilde{f}(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x}) dr - \int_0^s \frac{1}{2} a^2 e^{aY_r^{t,x}} |Z_r^{t,x}|^2 dr - \int_0^s ae^{aY_r^{t,x}} Z_r^{t,x} \downarrow dW_r \\ & \leq e^{a\Phi(X_0^{t,x})} + \int_0^s \left( -\frac{1}{2} a^2 + |a|K \right) e^{aY_r^{t,x}} |Y_r^{t,x}|^2 dr + \int_0^s |a|K e^{aY_r^{t,x}} dr - \int_0^s ae^{aY_r^{t,x}} Z_r^{t,x} \downarrow dW_r. \end{aligned}$$

Let  $\zeta(\omega') := \operatorname{esssup}_{[0, t] \times \Omega''} |Y_s^{t,x}(\omega', \omega'')| (< +\infty, \omega' \in \tilde{\Omega}')$ . Hence, by taking the conditional expectations  $\mathbb{E}[\cdot | \mathcal{H}_t]$  on both sides, we deduce that for any  $\mathbb{H}$ -stopping times  $\tau \in [0, T]$ ,

$$\begin{aligned} & \left( \frac{1}{2} a^2 - |a|K \right) e^{-|a|\zeta} \mathbb{E} \left[ \int_0^\tau |Z_r^{t,x}|^2 dr \middle| \mathcal{H}_\tau \right] \leq \left( \frac{1}{2} a^2 - |a|K \right) \mathbb{E} \left[ \int_0^\tau e^{aY_r^{t,x}} |Z_r|^2 dr \middle| \mathcal{H}_\tau \right] \\ & \leq \mathbb{E} \left[ e^{a\Phi(X_0^{t,x})} - e^{aY_\tau^{t,x}} + \int_0^\tau |a|K e^{aY_r^{t,x}} ds \middle| \mathcal{H}_\tau \right]. \end{aligned}$$

We can choose  $a = 4K$  such that  $\frac{1}{2}a^2 - |a|K = 4K^2$  and we get, keeping in mind that here  $K$  is a random variable bounded by  $\exp\{C \sup_{s \in [0, t]} |B_s|\}$  ( $C \in \mathbb{R}^+$  is a real constant, see Remark 4.4),

$$\mathbb{E} \left[ \int_0^\tau |Z_r^{t,x}|^2 dr \middle| \mathcal{H}_\tau \right] \leq \frac{(2 + 4K^2t)}{4K^2} e^{8K} \leq \exp \left\{ \exp \left\{ C \sup_{s \in [0, t]} |B_s| \right\} \right\}. \quad (4.10)$$

**Step 3.** Let us show that the process  $(Y^{t,x}, Z^{t,x})$  is not only  $\mathbb{H}$ - but also  $\mathbb{G}$ -adapted. For this we consider for an arbitrarily given  $\tau \in [0, t]$  equation (4.7) over the time interval  $[0, \tau]$ :

$$Y_s^{t,x} = \Phi(X_0^{t,x}) + \int_0^s \tilde{f}(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x}) dr - \int_0^s Z_r^{t,x} \downarrow dW_r, \quad s \in [0, \tau]. \quad (4.11)$$

Let  $\mathcal{H}_t^\tau := \mathcal{F}_{t,T}^W \vee \mathcal{F}_\tau^B$ ,  $t \in [0, \tau]$ . Then  $\mathbb{H}^\tau = \{\mathcal{H}_t^\tau\}_{t \in [0, \tau]}$  is a backward Brownian filtration enlarged by a  $\sigma$ -algebra generated by the fBm  $B$ , which is independent of the Brownian filtration. Thus, with respect to  $\mathbb{H}^\tau$ , the Brownian motion  $W$  has the martingale representation property. Since  $\tilde{f}(r, x, y, z)$  is  $\mathcal{G}_t$ - and, hence, also  $\mathcal{H}_t^\tau$ -measurable,  $dr$  a.e. on  $[0, \tau]$ , it follows from the classical BSDE theory (or Theorem 4.5) that BSDE (4.11) admits a unique solution  $(Y^{t,x,\tau}, Z^{t,x,\tau}) \in \mathcal{H}_\tau^\infty(\mathbb{R}) \times \mathcal{H}_\tau^2(\mathbb{R}^d)$ . On the other hand, also  $(Y_r^{t,x,\tau}, Z_r^{t,x,\tau})_{r \in [0, \tau]}$  is a solution of (4.7). Hence,  $(Y_r^{t,x}, Z_r^{t,x}) = (Y_r^{t,x,\tau}, Z_r^{t,x,\tau})$ ,  $dr$  a.e., for  $t < \tau$ . Consequently,  $(Y_r^{t,x}, Z_r^{t,x})$  is  $\mathcal{H}_r^\tau$ -measurable,  $dr$  a.e., for  $r < \tau$ . Therefore, letting  $\tau \downarrow t$  we can deduce from the right continuity of the filtration  $\mathbb{F}^B$  that  $(Y^{t,x}, Z^{t,x})$  is  $\mathbb{G}$ -adapted. ■

**Remark 4.8.** We remind the reader that the bound we get in (4.10) is only  $\mathbb{P}$ -a.s. finite, but not square-integrable. As a matter of fact, it is hard to prove directly that  $Z^{t,x}$  is a square-integrable process, which constitutes the main reason that we use instead the space  $\mathcal{H}_t^2(\mathbb{R}^d)$ . Hence, the major difference between our work and Buckdahn and Ma [5] is: In the classical case, a priori we can solve the BDSDE in the first step to get the square integrability of  $Z$ , but in the fractional case, there is not a direct way to solve the BDSDE.

Now we are ready to give the main result of this section by linking the BSDEs (4.2) and (4.7) with the help of the Doss-Sussman transformation.

**Theorem 4.9.** Let us define a new pair of processes  $(U^{t,x}, V^{t,x})$  by

$$U_s^{t,x} = \eta(s, Y_s^{t,x}), \quad V_s^{t,x} = \frac{\partial}{\partial y} \eta(s, Y_s^{t,x}) Z_s^{t,x},$$

where  $(Y^{t,x}, Z^{t,x})$  is the solution of BSDE (4.7). Then  $(U^{t,x}, V^{t,x}) \in \mathcal{H}_t^\infty(\mathbb{R}) \times \mathcal{H}_t^2(\mathbb{R}^d)$  is the solution of BDSDE (4.2).

**Remark 4.10.** The above theorem can be considered as a counterpart of Theorem 3.9 in Jing and León [11] for the semi-linear case when  $H < 1/2$ . However, since we use here a different Hurst parameter  $H$  and a different type of stochastic integral with respect to fBm  $B$ , the above theorem obviously does not cover the result in [11].

*Proof of Theorem 4.9:* The fact that  $(U^{t,x}, V^{t,x}) \in \mathcal{H}_t^\infty(\mathbb{R}) \times \mathcal{H}_t^2(\mathbb{R}^d)$  follows directly from Theorem 4.7 and Lemma 4.2. In order to prove the remaining part of the theorem, we just have to apply the Itô formula (Theorem 3.1) to  $\alpha(Y_s^{t,x}, B_s)$ , noticing  $\alpha(Y_s^{t,x}, B_s) = \eta(s, Y_s^{t,x})$ , to obtain that, for  $(s, x) \in [0, t] \times \mathbb{R}^d$ , the Russo-Vallois integral  $\int_0^s g(U_r^{t,x}) dB_r$  exists and

$$\begin{aligned} U_s^{t,x} = & \Phi_0(X_0^{t,x}) + \int_0^s \frac{\partial}{\partial y} \alpha(Y_r^{t,x}, B_r) \left[ \left( \frac{\partial}{\partial y} \eta(r, Y_r^{t,x}) \right)^{-1} \left\{ f \left( r, X_r^{t,x}, \eta(s, Y_r^{t,x}), \frac{\partial}{\partial y} \eta(r, Y_r^{t,x}) Z_r^{t,x} \right) \right. \right. \\ & + \frac{1}{2} \text{tr} \left[ (Z_r^{t,x})^T \frac{\partial^2}{\partial y^2} \eta(r, Y_r^{t,x}) Z_r^{t,x} \right] \Big\} dr - Z_r^{t,x} \downarrow dW_r \Big] - \frac{1}{2} \int_0^s \text{tr} \left[ (Z_r^{t,x})^T \frac{\partial^2}{\partial y^2} \eta(r, Y_r^{t,x}) Z_r^{t,x} \right] dr \\ & + \int_0^s \frac{\partial}{\partial z} \alpha(Y_r^{t,x}, B_r) dB_r. \end{aligned} \quad (4.12)$$

Consequently,

$$U_s^{t,x} = \Phi(X_0^{t,x}) + \int_0^s f(r, X_r^{t,x}, U_r^{t,x}, V_r^{t,x}) dr + \int_0^s g(U_r^{t,x}) dB_r - \int_0^s V_r^{t,x} \downarrow dW_r. \quad (4.13)$$

The proof is complete.  $\blacksquare$

Now we close this section by a simple example to illustrate the idea of the above procedure.

**Example 4.11.** *Let us consider the linear case ( for simplicity of notations we omit the superscript  $(t, x)$ ):*

$$\begin{cases} dU_s = U_s dB_s + f(U_s, V_s) ds + V_s \downarrow dW_s, & s \in [0, t]; \\ U_0 = \Phi(X_0^{t,x}). \end{cases} \quad (4.14)$$

*It is elementary to show that  $\alpha(y, z) = ye^z$ . Hence, the solution  $(U, V)$  of equation (4.14) is given by  $U_s = Y_s e^{B_s}$  and  $V_s = Z_s e^{B_s}$ , where  $(Y, Z)$  is the solution of*

$$\begin{cases} dY_s = \hat{f}(Y_s, Z_s) ds + Z_s \downarrow dW_s, & s \in [0, t]; \\ Y_0 = \Phi(X_0^{t,x}). \end{cases} \quad (4.15)$$

*and the function  $\hat{f}$  is defined by  $\hat{f}(y, z) = e^{-B_t} f(ye^{B_t}, ze^{B_t})$ . Obviously  $\hat{f}$  is Lipschitz in  $(y, z)$  as far as  $f$  is Lipschitz. By following a classical Malliavin calculus method (see, e.g., Pardoux and Peng [16]), we can get that  $Z$  is  $\mathbb{P}$ -a.s. uniformly bounded and, thus, the process  $\{Y_s, s \in [0, t]\}$  is  $(1/2 - \varepsilon)$ -Hölder continuous in  $s$ , for all  $\varepsilon \in (0, 1/2)$ . Therefore, we have, for every  $r, s \in [0, t]$ ,*

$$\begin{aligned} |U_r - U_s| &= |Y_r e^{B_r} - Y_s e^{B_s}| \leq \|Y\|_\infty |e^{B_r} - e^{B_s}| + e^{|B_s|} |Y_r - Y_s| \\ &\leq \exp \left\{ \sup_{0 \leq u \leq t} |B_u| \right\} \|Y\|_\infty C(\omega') |r - s|^\alpha + \exp \left\{ \sup_{0 \leq u \leq t} |B_u| \right\} C(\omega') |r - s|^{1/2 - \varepsilon}, \text{ for a.a. } \omega' \in \Omega', \end{aligned}$$

*where we can choose  $\alpha$  to be in  $(1/2, H)$ . That is to say, we can choose  $0 < \varepsilon < \alpha - 1/2$  and get that the process  $\{U_s, s \in [0, t]\}$  has  $\alpha$ -Hölder continuous paths. Hence, instead of using the Russo-Vallois integral with respect to the fractional Brownian motion in equation (4.14) we can use the classical Young integral. Furthermore, thanks to Proposition 2.3, these two integrals coincide.*

**Remark 4.12.** In the latter example, the Young integral and the Russo-Vallois integral coincide. Unfortunately, the Hölder continuity seems to be very hard to deduce in the general, nonlinear case. As a consequence, we have to work with the more general Russo-Vallois integral.

## 5 Associated Stochastic Partial Differential Equations

In this section we discuss briefly the relationship between an associated SPDE and a PDE with stochastic coefficients. For simplicity, we only show the relationship for the case of classical solutions. For a complete discussion of the case of (stochastic) viscosity solutions, one can proceed by adapting the approaches in Buckdahn and Ma [5] and Jing and León [11].

Let  $\mathcal{L}$  be the second order elliptic differential operator:

$$\mathcal{L} = \frac{1}{2} \sum_{i,j=1}^n (\sigma \sigma^T)_{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^n b_i(x) \frac{\partial}{\partial x_i},$$

which means that it is the infinitesimal generator of the Markovian process  $\{X_s^{t,x}, s \in [0, t]\}$  defined by equation (4.1).

Our aim is to study the following semilinear SPDE driven by the fBm  $B$ :

$$\begin{cases} dv(t, x) = [\mathcal{L}v(t, x) - f(t, x, v(t, x), \sigma(x)^T \frac{\partial}{\partial x} v(t, x))] dt + g(v(t, x)) dB_t, & (t, x) \in (0, T) \times \mathbb{R}^n; \\ v(0, x) = \Phi(x), & x \in \mathbb{R}^n. \end{cases} \quad (5.1)$$

In the case that  $B$  is a Brownian motion (an fBm with Hurst parameter  $H = 1/2$ ), it is well known (see, Pardoux and Peng [16], Buckdahn and Ma [5]) that the random field  $v(t, x) := U_t^{t, x}$  solves (5.1) (in the viscosity sense if the coefficients are Lipschitz, and in the classical sense if the coefficients are  $C_b^3$ ), where  $U^{t, x}$  is the solution of BDSDE (4.2). Thus, it is natural to raise the following question: Can we also solve the SPDE (5.1) driven by the fBm  $B$ , by studying the properties of the solution of its associated BDSDE (4.2)? The answer is positive. Indeed, by applying the Doss-Sussman transformation, we will show that the PDE:

$$\begin{cases} du(t, x) &= \left( \mathcal{L}u(t, x) - \tilde{f}(t, x, u(t, x), \sigma(x)^T \frac{\partial}{\partial x} u(t, x)) \right) dt, & (t, x) \in (0, T) \times \mathbb{R}^n; \\ u(0, x) &= \Phi(x), & x \in \mathbb{R}^n, \end{cases} \quad (5.2)$$

is transformed into SPDE (5.1), where we recall that

$$\tilde{f}(t, x, y, z) = \left( \frac{\partial}{\partial y} \eta(t, y) \right)^{-1} \left\{ f(t, x, \eta(t, y), \frac{\partial}{\partial y} \eta(t, y) z) + \frac{1}{2} \text{Tr} \left[ z \frac{\partial^2}{\partial y^2} \eta(t, y) z^T \right] \right\}.$$

We also observe that following a similar argument as in Kobylanski [12], under some smoothness assumptions,  $u(t, x) := Y_t^{t, x}$ ,  $(t, x) \in [0, T] \times \mathbb{R}^n$  is the solution of equation (5.2), where  $Y^{t, x}$  is the solution of BSDE (4.7).

First we give the definition for the classical solutions of equations (5.1) and (5.2).

**Definition 5.1.** *We say a stochastic field  $w : \Omega' \times [0, T] \times \mathbb{R}^n \rightarrow \mathbb{R}$  is a classical solution of equation (5.1) (resp., (5.2)), if  $w \in C_{\mathbb{F}^B}^{0,2}$  and satisfies equation (5.1) (resp., (5.2)).*

We have the following proposition.

**Proposition 5.2.** *Suppose that  $u$  is a classical solution of equation (5.2), then  $\hat{u}(t, x) \triangleq \eta(t, u(t, x)) = \alpha(u(t, x), B_t)$  is a classical solution of SPDE (5.1). The converse holds also true: Every classical solution  $\hat{u}$  of equation (5.1) defines a classical solution  $u(t, x) = \mathcal{E}(t, \hat{u}(t, x))$  of equation (5.2).*

*Proof:* The claim that  $\hat{u}(t, x) \in C_{\mathbb{F}^B}^{0,2}$  follows from the regularity property of the functions  $\alpha$  and  $u$  and the fact that  $u$  is  $\mathbb{F}^B$ -adapted. Moreover, we first observe that  $\hat{u}(0, x) = \eta(0, u(0, x)) = u(0, x) = \Phi(x)$ , and we apply the Itô formula to  $\hat{u}(t, x)$  to obtain:

$$\begin{aligned} d\hat{u}(t, x) &= \frac{\partial}{\partial y} \alpha(u(t, x), B_t) \left( \mathcal{L}u(t, x) - \tilde{f} \left( t, x, u(t, x), \sigma(x)^T \frac{\partial}{\partial x} u(t, x) \right) \right) dt + \frac{\partial}{\partial z} \alpha(u(t, x), B_t) dB_t \\ &= \left[ \frac{\partial}{\partial y} \alpha(u(t, x), B_t) \mathcal{L}u(t, x) - f \left( t, x, \eta(t, u(t, x)), \frac{\partial}{\partial y} \eta(t, u(t, x)) \sigma(x)^T \frac{\partial}{\partial x} u(t, x) \right) \right. \\ &\quad \left. - \frac{1}{2} \text{Tr} \left[ \sigma(x)^T \frac{\partial}{\partial x} u(t, x) \frac{\partial^2}{\partial y^2} \eta(t, u(t, x)) \frac{\partial}{\partial x} u(t, x)^T \sigma(x) \right] \right] dt + \frac{\partial}{\partial z} \alpha(u(t, x), B_t) dB_t. \end{aligned} \quad (5.3)$$

We notice that

$$\begin{aligned} \frac{\partial}{\partial x} \hat{u}(t, x) &= \frac{\partial}{\partial x} [\alpha(u(t, x), B_t)] = \left( \frac{\partial}{\partial y} \alpha \right) (u(t, x), B_t) \frac{\partial}{\partial x} u(t, x), \\ \frac{\partial^2}{\partial x^2} \hat{u}(t, x) &= \frac{\partial}{\partial x} u(t, x) \left( \frac{\partial^2}{\partial y^2} \alpha \right) (u(t, x), B_t) \left( \frac{\partial}{\partial x} u(t, x) \right)^T + \left( \frac{\partial}{\partial y} \alpha \right) (u(t, x), B_t) \frac{\partial^2}{\partial x^2} u(t, x). \end{aligned}$$

Thus, by substituting the above relations into (5.3), we can simplify the equation (5.3) and we get

$$d\hat{u}(t, x) = \left[ \mathcal{L}\hat{u}(t, x) - f \left( t, x, \hat{u}(t, x), \sigma(x)^T \frac{\partial}{\partial x} \hat{u}(t, x) \right) \right] dt + g(\hat{u}(t, x)) dB_t. \quad (5.4)$$

---

<sup>2</sup> $C_{\mathbb{F}^B}^{0,2}$  is the space of  $\mathbb{F}^B$ -adapted processes  $u(t, x)$  which are continuous in  $t$  and  $C^2$  in  $x$ .

Consequently,  $\hat{u}$  is a solution of SPDE (5.1). The proof of the converse direction is analogous. The proof is complete.  $\blacksquare$

In order to summarize, we have constructed a solution of SPDE (5.1) with the help of the fractional BDSDE (4.2), by passing through the quadratic BSDE (4.7) and the associated PDE (5.2) with random coefficients:

$$\begin{array}{ccc} \text{fract. BDSDE (4.2)} & \xRightarrow{\text{Thm. 4.9}} & \text{quadratic BSDE (4.7)} \\ & & \downarrow \\ \text{fract. SPDE (5.1)} & \xLeftrightarrow{\text{Prop. 5.2}} & \text{PDE (5.2) (with random coefficients).} \end{array}$$

Proposition 5.2 shows that, in the case of a classical solution, the Doss-Sussman transformation establishes a link between PDE (5.2) and SPDE (5.1). This motivates us to give the following definition of the stochastic viscosity solution. For the cases  $H \in (0, 1/2]$ , the reader is referred to Buckdahn and Ma [5], Jing and León [11]. For the classical definition of the viscosity solution, we refer to Crandall et al. [6].

**Definition 5.3.** A continuous random field  $\hat{u} : \Omega' \times [0, T] \times \mathbb{R}^n \rightarrow \mathbb{R}$  is called a (stochastic) viscosity solution of equation (5.1) if and only if  $u(t, x) = \mathcal{E}(t, \hat{u}(t, x))$ ,  $(t, x) \in [0, T] \times \mathbb{R}^n$  is the viscosity solution of equation (5.2).

By following a similar argument as that developed in the proof of Theorem 4.9 in Jing and León [11], our preceding discussion leads to the following theorem:

**Theorem 5.4.** The stochastic field  $\hat{u} : \Omega' \times [0, T] \times \mathbb{R}^n \rightarrow \mathbb{R}$  defined by  $\hat{u}(t, x) \triangleq \alpha(Y_t^{t,x}, B_t)$  is a (stochastic) viscosity solution of SPDE (5.1).

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